

A. Cohomology of BU_n .

$$\boxed{\eta_n}$$

↓ topological/modular vector bundle

$$\underline{BU_n} \simeq \underline{Gr_n(\mathbb{C}^\infty)}$$

$$c_i = \boxed{c_i(\eta_n)} \in H^{2i}(BU_n, \mathbb{Z}).$$

Then. $H^*(BU_n, \mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_n], |c_i| = 2i.$

Proof. $P: \boxed{(BU_1)^{\times n}} \xrightarrow{\text{classifies}} BU_n$
 $\downarrow \text{fib}$ $\downarrow \text{fib}$
 $(\mathbb{C}P^\infty)^{\times n}$ $\eta_1 \oplus \dots \oplus \eta_1 = \bigoplus_{i=1}^n P_i^+ \eta_1$
 n times



$$\underline{H^*(BU_1^{\times n}, \mathbb{Z})} \cong \underline{\mathbb{Z}[x_1, \dots, x_n]},$$

Künneth

$$P^+(c_1) = x_1 + \dots + x_n$$

$$|x_i| = 2,$$

⋮

$$P^+(c_n) = x_1 \cdots x_n$$

$$x_i = c_i(\eta_1) = c_i(P_i^+ \eta_1)$$

Also, every elmt in $P^*(H^*(BU_n, \mathbb{Z}))$ is symmetric since parity doesn't change $\eta_1^{\oplus n}$.

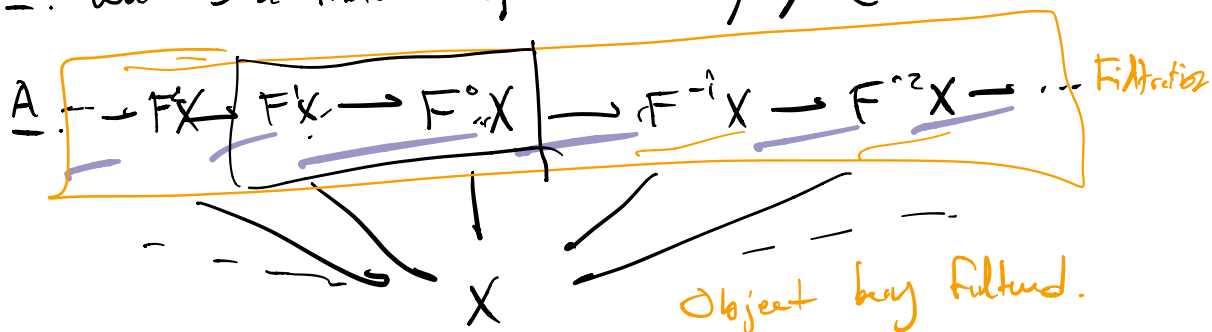
$$P^+(c_i) = c_i(\eta_1^{\oplus n}) = \text{ith}$$

elementary symmetric function in the x_i .

Now, enough to show injectivity. Do this via Schubert cells or a s.s.

B. Filtrations.

Q. What is a ^(decreasing) filtered object in a category \mathcal{C} ?



$$\mathbb{Z}^{op} : \left(\cdots \rightarrow 2 \rightarrow 1 \rightarrow 0 \rightarrow -1 \rightarrow -2 \rightarrow \cdots \right)$$

decreasing

Def. The category of filtrations in \mathcal{C}

is $\mathcal{F}\mathcal{C} = \text{Fun}(\mathbb{Z}^{op}, \mathcal{C})$. The

category of filtered objects in \mathcal{C}

is the category of pairs $(F^* X, X)$

of $F^* X \in \mathcal{F}\mathcal{C}$, $X \in \mathcal{C}$ and a

commutative diagram as above. F^∞

Def. (i) A filtration $F^* X$ is complete if $\lim_{n \rightarrow \infty} F^n X = 0$.

(ii) A filtered object is exhaustive if

$$F^{-\infty} := \text{colim}_{n \rightarrow -\infty} F^n X \cong X.$$

Ex. $FAb =$ filtrations in abelian groups. FAb

(a) $p^* \mathbb{Z} : p^n \mathbb{Z} = (p^n), n \geq 0.$

$F_{\text{in}}(\mathbb{Z}^{op}, Ab)$

Exhaustive Filtration on \mathbb{Z} .

Even $\mathbb{Z}_{\geq 0}^{op}$ -indexed.

$\dots \rightarrow p^2 \mathbb{Z} \rightarrow p \mathbb{Z} \rightarrow \mathbb{Z} = \mathbb{Z} = \mathbb{Z} \dots$
 $2 \quad 1 \quad 0 \quad -1 \quad -2$

Q. Is it complete?

A. $\lim_{n \rightarrow \infty} (p^n) = \bigcap_{n \geq 0} (p^n) = 0.$ (\mathbb{Z} is p -adically separated)

(b) We can rewrite the filtration above as

$\dots \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z} = \mathbb{Z} = \mathbb{Z} = \mathbb{Z} \dots$
 (curved arrow from \mathbb{Z} to \mathbb{Z} labeled p^2)

(c) $\dots \xrightarrow{p} \mathbb{Q}_p / \mathbb{Z}_p \xrightarrow{p} \mathbb{Q}_p / \mathbb{Z}_p \xrightarrow{p} \mathbb{Q}_p / \mathbb{Z}_p$

$\mathbb{Q}_p / \mathbb{Z}_p = \mathbb{Q} / \mathbb{Z} [p^{-\infty}]$
 $\cup_{n \geq 1} \mathbb{Z} / p^n$

Q. Is it complete?

A. No. An element of the limit is

$(a_n), a_n \in \mathbb{Q}_p / \mathbb{Z}_p, p a_n = a_{n-1}.$

C. Filtered complexes.

R a ring

$$F(D(R)) =: DF(R).$$

$D(FMod_R)$

This is the derived category of the abelian category of filtered R -modules.

Warning. Previously it was common to work only with strict filtrations in Mod_R , i.e., filtrations with each $F^i X \rightarrow F^{i-1} X$ injective. This is not an abelian category.

Filtered colimits are exact:

$$F^{-\infty}: \underline{FMod}_R \xrightarrow{\text{colim}} \underline{Mod}_R$$

is exact

$$F^{\infty}: FMod_R \xrightarrow{\text{lim}} Mod_R$$

is left exact.

Ex.

$f \in R$
non zero div.

$$\dots \rightarrow R \xrightarrow{f} R \rightarrow \dots \quad (1)$$

$$f \downarrow \quad \parallel$$

$$\dots \rightarrow R \xrightarrow{=} R \rightarrow \dots \quad (2)$$

$$\downarrow R/f \rightarrow 0$$

not strict!

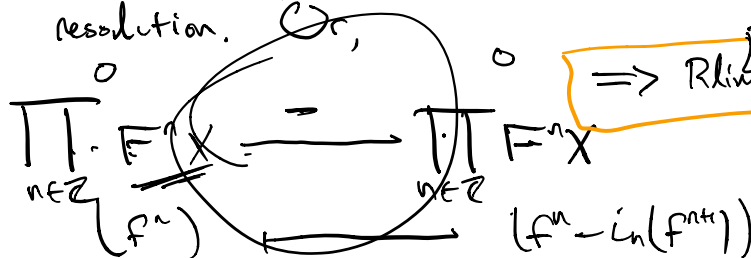
$$F^{-\infty}: DF(R) \xrightarrow{\text{colim}} D(R)$$

$$F^{\infty}: DF(R) \xrightarrow{R\text{lim}} D(R)$$

computed via injective resolution.

$$F^{n+1} X \xrightarrow{in} F^n X$$

$0, 1$
 $R\text{lim}_{n \in \mathbb{Z}} F^n X$
SI
cone $[-1]$



$$\Rightarrow R\text{lim}_{i \geq 1} F^i X = 0$$

Def. $F^*X \in DF(\mathbb{R})$ is complete

if $R\lim F^*X \simeq 0$.

Ex. $p^*\mathbb{Z} \in DF(\mathbb{Z})$

$$\boxed{p^*\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/p^*\mathbb{Z}}$$

Fiber seq of $DF(\mathbb{Z})$

$$\begin{array}{ccccccc}
 0 \rightarrow R\lim^0 p^*\mathbb{Z} & \rightarrow & R\lim^0 \mathbb{Z} & \rightarrow & R\lim^0 \mathbb{Z}/p^*\mathbb{Z} & \rightarrow & R\lim^1 p^*\mathbb{Z} \rightarrow R\lim^1 \mathbb{Z} \xrightarrow{\cong} 0 \\
 \text{exact} \parallel & & \parallel & & \parallel & & \parallel \\
 0 & & \mathbb{Z} & & \mathbb{Z}_p & & \mathbb{Z}/p\mathbb{Z} \\
 & & & & & & \downarrow \cong \\
 & & & & & & R\lim^1 \mathbb{Z}/p^*\mathbb{Z} \\
 & & & & & & \downarrow \\
 & & & & & & 0
 \end{array}$$

$$\begin{array}{ccccccc}
 \cdots \rightarrow p^2\mathbb{Z} \rightarrow p\mathbb{Z} \rightarrow \mathbb{Z} \xrightarrow{=} \mathbb{Z} \xrightarrow{=} \mathbb{Z} \rightarrow \cdots \\
 \downarrow \quad \downarrow \quad \downarrow \quad \parallel \quad \downarrow \\
 \cdots \rightarrow \mathbb{Z} = \mathbb{Z} = \mathbb{Z} = \mathbb{Z} = \mathbb{Z} \rightarrow \cdots \\
 \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
 \cdots \rightarrow \mathbb{Z}/p^2\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \cdots
 \end{array}$$

$\implies p^*\mathbb{Z}$ is not complete.

But, $p^*\mathbb{Z}_p$ is complete.