A. Cohomology of BU_n.

$$\boxed{\eta_n}$$
$$\downarrow \quad \text{tautological / modular vector bundle}$$
$$BU_n \simeq Gr_n(\mathbb{C}^\infty)$$

$$c_i = \boxed{c_i(\eta_n)} \in H^{2i}(BU_n, \mathbb{Z}).$$

Thm. $H^*(BU_n, \mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_n]$, $|c_i| = 2i$.

Proof. $p : \boxed{(BU_1)^{\times n}} \xrightarrow{\text{classifies}} BU_n$

$(BU_1)^{\times n} \simeq (\mathbb{C}P^\infty)^{\times n}$

$\underbrace{\eta_1 \oplus \cdots \oplus \eta_1}_{n \text{ times}} = \bigoplus_{i=1}^{n} p_i^* \eta_1$

$$BU_1^{\times n} \xrightarrow{p_i} BU_1$$

$H^*(BU_1^{\times n}, \mathbb{Z}) \underset{\text{Künneth}}{\cong} \mathbb{Z}[x_1, \dots, x_n]$,

$|x_i| = 2$,

$x_i = c_1(\eta_1) = c_1(p_i^* \eta_1)$

$p^*(c_1) = x_1 + \dots + x_n$

$\vdots$

$p^*(c_n) = x_1 \cdots x_n$

$p^*(c_i) = c_i(\eta_1^{\oplus n}) =$ ith elementary symmetric function in the x_i.

Also, every element in $p^*(H^*(BU_n, \mathbb{Z}))$ is symmetric since permuting doesn't change $\eta_1^{\oplus n}$.

Now, enough to show injectivity. Do this via Schubert cells or a s.s.

B. Filtrations.

Q. What is a (decreasing) filtered object in a category $\mathcal{C}$?

A.

$$\begin{tikzcd}
\cdots \arrow[r] & F^1X \arrow[r] \arrow[drr] & F^0X \arrow[r] \arrow[dr] & F^{-1}X \arrow[r] \arrow[d] & F^{-2}X \arrow[r] \arrow[dl] & \cdots \\
& & & X & &
\end{tikzcd}$$

Filtration

Object being filtered.

$$\mathbb{Z}^{op}: \left(\cdots \to 2 \to 1 \to 0 \to -1 \to -2 \to \cdots \right)$$

Def. The category of (decreasing) filtrations in $\mathcal{C}$ is $F\mathcal{C} = \mathrm{Fun}(\mathbb{Z}^{op}, \mathcal{C})$. The category of filtered objects in $\mathcal{C}$ is the category of pairs (F^*X, X) of $F^*X \in F\mathcal{C}$, $X \in \mathcal{C}$ and a commutative diagram as above.

Def. (i) A filtration F^*X is complete if $F^{\infty} = \lim_{n \to \infty} F^nX \simeq 0$.

(ii) A filtered object is exhaustive if
$$F^{-\infty} := \operatorname{colim}_{n \to -\infty} F^nX \xrightarrow{\sim} X.$$

Ex. FAb = filtrations in abelian groups. FAb $\subseteq$ Fun($\mathbb{Z}^{op}$, Ab)

(a) $p^*\mathbb{Z}$: $p^n\mathbb{Z} = (p^n)$, $n \geq 0$.

Exhaustive filtration on $\mathbb{Z}$.

Even $\boxed{\mathbb{Z}^{op}_{\geq 0}\text{-indexed.}}$

$$\cdots \to p^2\mathbb{Z} \to p\mathbb{Z} \to \mathbb{Z} = \mathbb{Z} = \mathbb{Z} \cdots$$

$$2 \qquad 1 \qquad 0 \qquad -1 \qquad -2$$

Q. Is it complete?

A. $\lim_{n\to\infty}(p^n) = \bigcap_{n\geq 0}(p^n) = 0$. ($\mathbb{Z}$ is p-adically separated)

(b) We can rewrite the filtration above as

$$\begin{tikzcd} \cdots \arrow[r, "p"] & \mathbb{Z} \arrow[r, "p"] \arrow[rr, bend right, "p^2"] & \mathbb{Z} \arrow[r, "p"] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \cdots \end{tikzcd}$$

(c) $\cdots \xrightarrow{p} \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{p} \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{p} \mathbb{Q}_p/\mathbb{Z}_p$

$$\mathbb{Q}_p/\mathbb{Z}_p = \mathbb{Q}/\mathbb{Z}[p^\infty] = \bigcup_{n\geq 1} \mathbb{Z}/p^n$$

Q. Is it complete?

A. No. An element of the limit is

(a_n), $a_n \in \mathbb{Q}_p/\mathbb{Z}_p$, $pa_n = a_{n-1}$.

C. Filtered complexes.

R a ring

$$F(D(R)) =: DF(R).$$

$$\overset{\shortparallel}{D(FMod_R)}$$

This is the derived category of the abelian category of filtered R-modules.

Warning. Previously it was common to work only with strict filtrations in Mod_R, i.e., filtrations with each $F^iX \to F^{i-1}X$ injective. This is not an abelian category.

Filtered colimits are exact:

$F^{-\infty}$: $FMod_R \xrightarrow{colim} Mod_R$ is exact

F^{∞}: $FMod_R \xrightarrow{lim} Mod_R$ is left exact.

Ex. $f \in R$ nonzero div.

$$\begin{tikzcd}
\cdots \arrow[r] & R \arrow[r, "f"] \arrow[d, "f"'] & R \arrow[r] \arrow[d, "\shortparallel"] & \cdots & (1) \\
\cdots \arrow[r] & R \arrow[r, "="] \arrow[d] & R \arrow[r] \arrow[d] & \cdots & (2) \\
& R/f \arrow[r] & 0 & &
\end{tikzcd}$$

not strict!

$$F^{-\infty}: DF(R) \xrightarrow{colim} D(R)$$

$$F^{\infty}: DF(R) \xrightarrow{R\lim} D(R)$$

computed via injective resolution.

$$F^{n+1}X \xrightarrow{i_n} F^nX$$

$$\overset{0,1}{R\lim F^*X} \longrightarrow \overset{0}{\prod_{n\in\mathbb{Z}} F^nX} \longrightarrow \overset{0}{\prod_{n\in\mathbb{Z}} F^nX}$$

$$(f^n) \longmapsto (f^n - i_n(f^{n+1})).$$

$R\lim F^*X \cong cone[-1]$

$$\boxed{\Rightarrow R\lim^{\boxed{i}} = 0 \quad \underline{i > 1}.}$$

Def. $F^\star X \in DF(R)$ is complete

if $R\lim F^\star X \simeq 0$.

Ex. $p^\star \mathbb{Z} \in DF(\mathbb{Z})$

$$\boxed{p^\star\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/p^\star\mathbb{Z}}$$ fiber seq in $DF(\mathbb{Z})$

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & R\lim^0 p^\star\mathbb{Z} \arrow[r] \arrow[d, equal, "\text{earlier}"'] & R\lim^0 \mathbb{Z} \arrow[r] \arrow[d, equal] & R\lim^0 \mathbb{Z}/p^\star\mathbb{Z} \arrow[r] \arrow[d, "\cong"] & \boxed{R\lim^1 p^\star\mathbb{Z}} \arrow[r] \arrow[d, "\cong"] & R\lim^1\mathbb{Z} \arrow[d] \\
& 0 & \mathbb{Z} & \mathbb{Z}_p & \mathbb{Z}_p/\mathbb{Z} \arrow[d, phantom, "\not\simeq"] & R\lim^1 \mathbb{Z}/p^\star\mathbb{Z} \arrow[d] \\
& & & & 0 & 0
\end{tikzcd}$$

$R\lim^1\mathbb{Z} \simeq 0$, $R\lim^1 \mathbb{Z}/p^\star\mathbb{Z} \simeq 0$

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & p^2\mathbb{Z} \arrow[r] \arrow[d] & p\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, equal] \arrow[d] & \mathbb{Z} \arrow[r, equal] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & \mathbb{Z} \arrow[r, equal] \arrow[d] & \mathbb{Z} \arrow[r, equal] \arrow[d] & \mathbb{Z} \arrow[r, equal] \arrow[d] & \mathbb{Z} \arrow[r, equal] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathbb{Z}/p^2 \arrow[r] & \mathbb{Z}/p \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

$\Longrightarrow$ $p^\star\mathbb{Z}$ is <u>not</u> complete.

But, $p^\star\mathbb{Z}_p$ <u>is</u> complete.